Levy's inversion: X cts, pdf f_X c.f. ϕ_X,

then $f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t)\, dt$ (inv-Fourier-transf)

e.g.: (5.7.9, Parseval Equality)

X has pdf f_X, c.f. ϕ_X, show that if f is regular enough,

$$\int_{\mathbb{R}} f_X^2(x)\, dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\phi(t)|^2\, dt$$

Pf: $|\phi_X(t)|^2 = \phi_X(t) \cdot \overline{\phi_X(t)} = \mathbb{E} e^{itX} \cdot \mathbb{E} e^{-itX}$

$= \mathbb{E} e^{itX} \cdot \mathbb{E} e^{-it\tilde{X}}$

where $\tilde{X} \overset{d}{=} X$, $\tilde{X}$ indep of X

$= \mathbb{E} e^{it(X - \tilde{X})}$

So $|\phi_X(t)|^2 = \phi_{X-\tilde{X}}(t)$.

By Levy's inversion thm, $f_{X-\tilde{X}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \cdot |\phi_X(t)|^2\, dt$

convolution $\|$

$\int_{\mathbb{R}} f_X(y) \cdot f_{\tilde{X}}(y-x)\, dy$

$\|$

$\int_{\mathbb{R}} f_X(y) \cdot f_X(y-x)\, dy$

Set $x=0$: $\int_{\mathbb{R}} f_X^2(y)\,dy = \frac{1}{2\pi}\int_{\mathbb{R}} |\phi_X(t)|^2\,dt$ ✓

Fourier transform as $\mathcal{L}^2$-isometry

Levy's Continuity Thm:

If $\phi_n(t) \to \phi(t)$ pointwisely on $\mathbb{R}$, and ϕ is cts at $t=0$, then ϕ is c.f. and $F_n \xrightarrow{w} F$.

very useful to prove limiting distribution of a sum of independent r.v.

e.g.: $X \sim \Gamma(s,1)$, $Y|_{X=x} \sim \mathcal{P}(x)$, show

$$\frac{Y-\mathbb{E}Y}{\sqrt{\mathrm{Var}(Y)}} \xrightarrow{d} N(0,1) \qquad (s\to\infty)$$

Pf:

$$\phi_Y(t) = \mathbb{E}e^{itY} = \mathbb{E}\left[\mathbb{E}(e^{itY}|X)\right]$$

here $Y|X$ is Poisson, so $\mathbb{E}(e^{itY}|X) = e^{X(e^{it}-1)}$

Poisson c.f.

So $\phi_Y(t) = \mathbb{E}\,e^{X(e^{it}-1)} = \phi_X\left(\frac{e^{it}-1}{i}\right)$

Now $\phi_X(t) = \int_0^{+\infty} e^{itx}\cdot \frac{1}{\Gamma(s)} x^{s-1} e^{-x}\,dx$

$= (1-it)^{-s}$ (sum of s i.i.d. $\mathcal{E}(1)$ r.v.)

$\phi_Y(t) = (2-e^{it})^{-s}$, now that

$\phi_Y'(0) = i \cdot \mathbb{E}Y = si$, $\mathbb{E}Y = s$,

$\phi_Y''(0) = -\mathbb{E}Y^2 = -(s+2)s$, $\mathbb{E}Y^2 = s^2+2s$

$\mathrm{Var}(Y) = 2s$.

Now $Z = \frac{Y-s}{\sqrt{2s}}$, $\phi_Z(t) = \mathbb{E}e^{it\frac{Y-s}{\sqrt{2s}}}$

$$= e^{-it\sqrt{\frac{s}{2}}} \cdot \phi_Y\left(\frac{t}{\sqrt{2s}}\right)$$

$$= e^{-it\sqrt{\frac{s}{2}}} \cdot \left(2-e^{i\frac{t}{\sqrt{2s}}}\right)^{-s}$$

$$\log\phi_Z(t) = -it\cdot\sqrt{\frac{s}{2}} - s\cdot\log\left(2-e^{\frac{it}{\sqrt{2s}}}\right)$$

$$= -it\sqrt{\frac{s}{2}} - s\cdot\left[\left(1-e^{\frac{it}{\sqrt{2s}}}\right) - \frac{1}{2}\left(1-e^{\frac{it}{\sqrt{2s}}}\right)^2 + o\left(\left(1-e^{\frac{it}{\sqrt{2s}}}\right)^3\right)\right]$$

$(s\to\infty)$

$$= -it\sqrt{\frac{s}{2}} - s\cdot\left[-\frac{it}{\sqrt{2s}} - \frac{i^2t^2}{4s} - \frac{1}{2}\cdot\frac{i^2t^2}{2s} + o\left(\frac{1}{s}\right)\right]$$

$$= -\frac{t^2}{2} + o(1) \quad (s\to\infty)$$

So: $\phi_Z(t) \to e^{-\frac{t^2}{2}}$ $(s\to\infty)$, by Levy's

Continuity Thm, since $e^{-\frac{t^2}{2}}$ cts at 0,

$$Z \xrightarrow{d} N(0,1) \quad (s \to \infty)$$ ✓

Consistent with CLT due to $\Gamma(\alpha,\beta)$ being infinitely divisible for fixed β, and $\gamma(\lambda)$ is infinitely divisible.

<u>Application of CLT:</u>

eg: (5.10.1) For $x \geq 0$, as $n \to \infty$, show

(a): $$\sum_{k:|k-\frac{n}{2}| \leq \frac{x\sqrt{n}}{2}} \binom{n}{k} \sim 2^n \cdot \underbrace{\int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du}_{\mathbb{P}(|G| \leq x),\ G \sim N(0,1)} \quad (n \to \infty)$$

Consider $X_1, \ldots, X_n \sim B(1, \frac{1}{2})$ so $S_n = \sum_{i=1}^{n} X_i \sim B(n, \frac{1}{2})$

$\mathbb{E}S_n = \frac{n}{2}$, $Var(S_n) = \frac{n}{4}$, by CLT,

$$\frac{S_n - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \xrightarrow{d} N(0,1) \quad (n \to \infty)$$

Interpret $\displaystyle\sum_{k:|k-\frac{n}{2}|\le\frac{x\sqrt{n}}{2}} \binom{n}{k}\cdot\left(\frac{1}{2}\right)^n$ as

$$= \sum_{k:|k-\frac{n}{2}|\le\frac{x\sqrt{n}}{2}} \mathbb{P}(S_n=k)$$

$$= \mathbb{P}\left(|S_n-\mathbb{E}S_n|\le x\cdot\sqrt{\mathrm{Var}(S_n)}\right)$$

concludes the proof. ✓

(b): $\displaystyle\sum_{k:|k-n|\le x\sqrt{n}} \frac{n^k}{k!} \sim e^n\cdot\int_{-x}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \quad (n\to\infty)$

Similarly, consider $X_1,\dots,X_n \overset{\text{i.i.d.}}{\sim} \mathcal{P}(1)$, so $S_n\sim\mathcal{P}(n)$

$\dfrac{S_n-n}{\sqrt{n}} \xrightarrow{d} N(0,1) \quad (n\to\infty)$,

interpret $\displaystyle\sum_{k:|k-n|\le x\sqrt{n}} \frac{n^k}{k!}e^{-n}$

$$= \mathbb{P}\left(|S_n-\mathbb{E}S_n|\le x\cdot\sqrt{\mathrm{Var}(S_n)}\right)$$

concludes the proof. ✓

e.g.: (5.10.7, 5.10.8) $\frac{1}{2}$-stable law

$X_1, X_2, \cdots$ i.i.d. with dist $f(x) = \sqrt{2\pi x^{-3}} \cdot e^{-\frac{1}{2x}}$ $(x>0)$

its c.f. given by $\phi_X(t) = \begin{cases} e^{-(1-i)\sqrt{t}} & \text{if } t \geq 0 \\ e^{-(1+i)\sqrt{|t|}} & \text{if } t \leq 0 \end{cases}$,

let $U_n \triangleq \frac{1}{n} \sum_{r=1}^{n} X_r$, $T_n \triangleq \frac{1}{n} U_n$, show:

(a): $\mathbb{P}(U_n < c) \to 0$ for $\forall c < \infty$ as $n \to \infty$.

(b): $T_n \overset{d}{=} X_1$.

Pf:

(a): $\phi_{U_n}(t) = \mathbb{E} e^{it \cdot \frac{1}{n} \sum_{r=1}^{n} X_r} = \left[\phi_{X_1}\left(\frac{t}{n}\right)\right]^n$

$= \phi_{X_1}(nt) = \phi_{nX_1}(t)$

So $U_n \overset{d}{=} n X_1$.

$\mathbb{P}(U_n < c) = \mathbb{P}(X_1 < \frac{c}{n}) \to 0 \quad (n \to \infty)$

(b): $\phi_{T_n}(t) = \phi_{U_n}\left(\frac{t}{n}\right) = \phi_{X_1}(t)$, so $T_n \overset{d}{=} X_1$.

Example of failure of LLN.